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# Profinite groups, profinite completions and a conjecture of Moore

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## Abstract

Let  $R$  be any ring (with 1),  $\Gamma$  a group and  $R\Gamma$  the corresponding group ring. Let  $H$  be a subgroup of  $\Gamma$  of finite index. Let  $M$  be an  $R\Gamma$ -module, whose restriction to  $RH$  is projective.

Moore's conjecture (J. Pure Appl. Algebra 7(1976)287): Assume for every nontrivial element  $x$  in  $\Gamma$ , at least one of the following two conditions holds:

- (M1)  $\langle x \rangle \cap H \neq \{e\}$  (in particular this holds if  $\Gamma$  is torsion free)
- (M2)  $\text{ord}(x)$  is finite and invertible in  $R$ .

Then  $M$  is projective as an  $R\Gamma$ -module.

More generally, the conjecture has been formulated for crossed products  $R * \Gamma$  and even for strongly graded rings  $R(\Gamma)$ . We prove the conjecture for new families of groups, in particular for groups whose profinite completion is torsion free.

The conjecture can be formulated for profinite modules  $M$  over complete groups rings  $[[R\Gamma]]$  where  $R$  is a profinite ring and  $\Gamma$  a profinite group. We prove the conjecture for arbitrary profinite groups. This implies Serre's theorem on cohomological dimension of profinite groups.  
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## 1. Introduction

Let  $\Gamma$  be a group and  $H$  a subgroup of finite index. Let  $\mathbb{Z}\Gamma$  be the integral group ring and let  $M$  be a  $\mathbb{Z}\Gamma$ -module. It is well known (and easy to see) that if  $M$  is projective over  $\mathbb{Z}\Gamma$  then it is projective as a module over the subring  $\mathbb{Z}H$  (in fact this is true for arbitrary subgroups  $H$  of  $\Gamma$ ). Is the converse true? It is not difficult to construct examples which show that this is false. For instance let  $\Gamma$  be a nontrivial finite group,  $H = \{e\}$  and  $M \cong \mathbb{Z}$  with the trivial  $\Gamma$ -action. Clearly  $M$  is  $\mathbb{Z} = \mathbb{Z}H$ -free but not projective over  $\mathbb{Z}\Gamma$ . More generally, one can construct examples (that show that the converse is false) in which  $\Gamma$  contains a nontrivial element  $x$  of finite order such that  $\langle x \rangle \cap H = \{e\}$ .

Let  $R$  be an arbitrary ring with unit element 1 and let  $\Gamma$  be any group. Let  $R(\Gamma)$  be a strongly graded ring over  $R$  and  $M$  a module over  $R(\Gamma)$ . (By definition  $R(\Gamma)$  is strongly graded over  $R$  by  $\Gamma$  if  $R(\Gamma) \cong \bigoplus_{\sigma \in \Gamma} R_{\sigma}$ ,  $R_e \cong R$  and  $R_{\sigma}R_{\tau} = R_{\sigma\tau}$  for every  $\sigma, \tau$  in  $\Gamma$ .) Recall that if  $M$  is projective over  $R(\Gamma)$  then  $M$  is projective over the subring  $R(U)$  where  $U$  is any subgroup of  $\Gamma$ . Let  $H$  be a finite index subgroup of  $\Gamma$ .

**Moore's Conjecture** (see Aljadeff et al. [2, Conjecture 1.1]). Assume for every non-trivial element  $x$  in  $\Gamma$ , at least one of the following two conditions holds:

(M1)  $\langle x \rangle \cap H \neq \{e\}$

(M2)  $\text{ord}(x)$  is finite and invertible in  $R$ .

Then every  $R(\Gamma)$ -module  $M$  which is projective over  $R(H)$ , is projective also over  $R(\Gamma)$ .

We will say that Moore's conjecture holds for a group  $\Gamma$  if the conjecture above holds for an arbitrary ring  $R$  and an arbitrary subgroup of finite index  $H$  (see definitions in Section 1).

One of the main advantages of strongly graded rings (over group rings) is that they “allow induction”. Recall that if  $H$  is normal in  $\Gamma$  then the strongly graded ring  $R(\Gamma)$  can be expressed as a strongly graded ring over  $R(H)$  by  $\Gamma/H$ . (A similar result applies for crossed products but obviously not for group rings.) This “flexibility” will be used in our proofs.

### Remarks.

1. The conjecture was formulated for group rings in [6]. In [6, Proposition 8] it is shown that if Moore's conjecture holds for a class of groups  $C$ , then it holds for the class  $LC$  (locally  $C$ ) restricted to finitely generated modules. The main result in [6] implies that the conjecture holds for finite groups and hence the conjecture holds for locally finite groups restricted to finitely generated modules.
2. If  $R = \mathbb{Z}$  the ring of integers and  $\Gamma$  is torsion free, the conjecture says that every  $\mathbb{Z}\Gamma$ -module  $M$  which is  $\mathbb{Z}H$ -projective is also  $\mathbb{Z}\Gamma$ -projective.
3. Moore's conjecture is a far reaching generalization of Serre's theorem on cohomological dimension of groups [18].

**Serre’s Theorem.** *Let  $\Gamma$  be a group and  $H$  a subgroup of finite index. Assume  $H$  has finite cohomological dimension (that is  $\Gamma$  has virtual finite cohomological dimension). If  $\Gamma$  is torsion free then it has finite cohomological dimension. Moreover,  $cd(\Gamma) = cd(H)$ .*

Indeed, Moore’s conjecture implies Serre’s theorem as follows: let  $cd(H) = n$ . If  $P. \rightarrow \mathbb{Z} \rightarrow 0$  is a projective resolution of  $\mathbb{Z}$  over  $\mathbb{Z}\Gamma$ , it is projective also over  $H$ . It follows that the  $n$ th syzygy  $Y_n$  of the resolution is a  $\mathbb{Z}\Gamma$ -module whose restriction to  $H$  is projective. Moore’s conjecture says that  $Y_n$  is projective over  $\mathbb{Z}\Gamma$  and so  $cd(\Gamma) \leq n$ .

4. Moore’s conjecture holds for groups which belong to a certain class of groups  $H_1F$ . The class  $H_1F$  contains and is strictly larger than the class of groups of virtual finite cohomological dimension. If  $\Gamma \in H_1F$  and torsion free, then  $\Gamma$  has finite cohomological dimension. We refer the reader to [10, 2.1] for the precise definition of the class  $H_1F$ .
5. Kropholler has constructed also a much larger class of groups, denoted by  $HF$ . The class  $HF$  is extension closed, subgroup closed and closed under directed unions. In particular it contains
  - (a)  $H_1F$ ;
  - (b) every finitely generated soluble group;
  - (c) every countable linear group.

For this class one has

**Theorem** (Aljadeff et al. [2, Theorem 3.1]). *Let  $\Gamma \in HF$  and let  $R(\Gamma)$  be a strongly graded ring over  $R$ . Then Moore’s conjecture holds whenever the module  $M$  is finitely generated.*

Our first task is to “reduce” the problem to finitely generated groups (see also [6, Proposition 8]). More precisely

**Theorem 1.1.** *If Moore’s conjecture holds for every finitely generated subgroup of a group  $\Gamma$  then it holds for  $\Gamma$ .*

The proof is based on a result of Benson and Goodearl (see Section 2).

**Corollary** (Chouinard [6, Corollary 5.1]). *Moore’s conjecture holds for any abelian group.*

**Proof.** Every finitely generated abelian group belongs to  $H_1F$ .  $\square$

The main idea in this paper is to analyze the group  $\Gamma$  from its “top” rather than its “bottom” as in the construction of the classes  $H_1F$  and  $HF$ . By “top” we mean the finite quotients of  $\Gamma$  and by “bottom” we mean the subgroups of  $\Gamma$  that appear as “stabilizers” in Kropholler’s construction.

Let  $P(\Gamma)$  be the collection of all finite index, normal subgroups of  $\Gamma$ . Let  $\Omega_\Gamma$  be a subset of  $P(\Gamma)$  filtered from below. Assume further that  $\Omega_\Gamma$  is cofinal in  $P(\Gamma)$ . Denote by  $\widehat{\Gamma} = \lim_{\leftarrow} \Gamma/N$  the profinite completion of  $\Gamma$  with respect to  $\Omega_\Gamma$  and let  $\phi: \Gamma \rightarrow \widehat{\Gamma}$  be the canonical map induced by the natural projections  $\Gamma \rightarrow \Gamma/N$ ,  $N \in \Omega_\Gamma$ . Our main result (for discrete groups) is Theorem 3.1. Its formulation requires some terminology which is introduced in Section 2. Its main corollary is

**Theorem 1.2.** *Let  $\Gamma$  be a group and  $\widehat{\Gamma}$  its profinite completion as above. If  $\widehat{\Gamma}$  is torsion free then Moore's conjecture holds for  $\Gamma$ . More generally: If any element  $z$  of prime order (say  $p$ ) in  $\widehat{\Gamma}$  is conjugate to an element  $\phi(x)$  where  $x$  is an element (in  $\Gamma$ ) of order  $p$ , then Moore's conjecture holds for  $\Gamma$ .*

The condition in Theorem 1.2 is known to hold for large families of groups. For instance it holds for families such as (see [11])

1. soluble minimax groups (see [15] for the definition);
2. torsion free, finitely generated abelian by nilpotent groups.

**Remark.** The finitely generated condition in (2) is important. Indeed, in [11] an example is given of a residually finite, torsion-free abelian by nilpotent group with torsion in its profinite completion. Nevertheless, Moore's conjecture holds for such a group by Theorem 1.1.

It is not difficult to construct examples of finitely generated torsion-free metabelian groups with infinite cohomological dimension whose profinite completion is torsion free (e.g.  $\Gamma = \mathbb{Z}^\infty \wr \mathbb{Z} = \{((x_i)_{i \in \mathbb{Z}}, \sigma^j) \text{ where } \sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i-1})_{i \in \mathbb{Z}} \text{ (right shift)}\}$ ). Note that  $\Gamma$  does not belong to  $H_1 F$ .

The finitely generated groups considered so far belong to  $HF$ . It is known that the Thompson group

$$T = \langle x_0, x_1, x_2, \dots, x_n, \dots : x_n^{x_i} = x_{n+1} \text{ for every } i < n \rangle$$

is not in  $HF$ , (see [5,10]). On the other hand it is known that

1.  $\text{rad}(T) = \cap \{H : [\Gamma : H] < \infty\} = T'$ , the commutator subgroup of  $T$ .
2.  $T_{\text{ab}} = T/T'$  is free abelian of rank 2.

**Corollary 1.3.** *Moore's conjecture holds for the Thompson group.*

**Proof.**  $\widehat{T} \cong \widehat{\mathbb{Z}^2}$  is torsion free.  $\square$

It was an open question whether the profinite completion  $\widehat{\Gamma}$  of a torsion free, residually finite group  $\Gamma$  is necessarily torsion free. A first counterexample to this question was given by Evans [8]. Later, Lubotzky [12] gave an example of a torsion-free residually finite group whose profinite completion contains copies of any finite group!

Theorem 1.2 may be applied also to profinite groups viewed as discrete groups. Recently, Nikolov and Segal announced the following important result.

**Theorem** (Nikolov and Segal [13]). *Let  $\Gamma$  be a topologically finitely generated profinite group (this means that  $\Gamma$  has a f.g. subgroup which is dense in  $\Gamma$ ). Then every subgroup of finite index in  $\Gamma$  is open.*

It follows that  $\Gamma$  is naturally isomorphic to its profinite completion. In particular, the condition in Theorem 1.2 is satisfied by  $\Gamma$  and so we have the following:

**Corollary 1.4.** *Let  $\Gamma$  be a profinite group, topologically finitely generated. Then Moore’s conjecture holds for  $\Gamma$  viewed as a discrete group.*

It is easy to see that if in the proof of Theorem 1.1 we replace finitely generated subgroups by their closures in  $\Gamma$ , then we obtain a variant of Theorem 1.1 which together with Corollary 1.4 implies Moore’s conjecture for arbitrary profinite groups considered as discrete groups. We record this in

**Corollary 1.5.** *Let  $\Gamma$  be a profinite group. Then Moore’s conjecture holds for  $\Gamma$  viewed as a discrete group.*

The interest in Corollaries 1.4 and 1.5 is limited. It is of course desired to obtain similar results in the category of profinite modules. For this one needs to adapt Chouinard’s results in [6], to profinite rings and profinite modules. Let  $R$  be a profinite ring and  $\Gamma$  a profinite group. Denote by  $[[R\Gamma]]$  the complete group ring (see [16, Section 5.3]). Our main result in this context (and perhaps in the entire paper) is

**Theorem 1.6.** *Let  $[[R\Gamma]]$  be a complete group ring and  $M$  an  $[[R\Gamma]]$ -profinite module. Let  $H$  be an open subgroup of  $\Gamma$ . Assume Moore’s condition (M1) or (M2) holds. Then if  $M$  is projective over  $[[RH]]$  then  $M$  is projective over  $[[R\Gamma]]$ .*

This implies

**Corollary 1.7.** *Let  $[[R\Gamma]]$  be a complete group ring and  $H$  an open subgroup of  $\Gamma$ . Assume Moore’s condition (M1) or (M2) holds. Then for any  $[[R\Gamma]]$ -profinite module  $M$  we have  $\text{proj dim}_{[[R\Gamma]]}(M) = \text{proj dim}_{[[RH]]}(M)$ .*

As a direct consequence we get Serre’s theorem on cohomological dimensions of profinite groups.

**Theorem** (Serre [17], Haran [9]). *Let  $\Gamma$  be a profinite group and  $H$  an open subgroup. If  $\Gamma$  has no elements of order  $p$  then  $cd_p(\Gamma) = cd_p(H)$ .*

This paper is organized as follows: In Section 2 we set most of the terminology and notation needed in the paper. The section contains also two reductions one of which is the reduction to finitely generated subgroups mentioned above. The other reduction allows us to replace the subgroup of finite index  $H$  in Moore’s conjecture with its core in  $\Gamma$ , namely the intersection of all its conjugates in  $\Gamma$ . Next, we continue with

a brief discussion on the necessity of the condition in Moore's conjecture (basically the condition is necessary only if  $H$  is normal in  $\Gamma$ ). We close the section by recalling some basic facts on profinite topologies.

The main results are in Sections 3 and 4. In Section 3 we prove Theorem 1.2. As mentioned above it follows from Theorem 3.1 which contains the main construction in the paper. The last section, Section 4, contains the proof of Moore's conjecture for complete group rings and profinite modules.

## 2. Preliminaries, terminology and reductions

It is convenient to use the following terminology: if condition (M1) or (M2) holds for the group  $\Gamma$ , the subgroup of finite index  $H$  and the coefficient ring  $R$ , we will say that Moore's condition holds for the triple  $(\Gamma, H, R)$ . If (M1) holds for the group  $\Gamma$  and the subgroup  $H$  we will say that Moore's condition holds for the pair  $(\Gamma, H)$ . Note that (M1) holds for  $(\Gamma, H)$  if and only if (M1) or (M2) holds for  $(\Gamma, H, R)$ , where  $R$  is any ring. Next we will say that Moore's conjecture holds for the triple  $(\Gamma, H, R)$  if condition (M1) or (M2) implies that any module  $M$  over  $R(\Gamma)$  which is projective over  $R(H)$ , is projective over  $R(\Gamma)$ . We will say that Moore's conjecture holds for  $(\Gamma, H)$  if condition (M1) implies the same conclusion for  $(\Gamma, H, R)$  where  $R$  is arbitrary. Finally, we will say that Moore's conjecture holds for the group  $\Gamma$  if the conjecture holds for  $(\Gamma, H)$ , for any subgroup  $H$  of finite index in  $\Gamma$ .

We turn now to the proof of Theorem 1.1 starting with the following lemma which is a particular case of [7, p.125, exercise 17].

**Lemma 2.1.** *Let  $\Gamma$  be any group and  $R(\Gamma)$  a strongly graded ring over  $R$ . Let  $M$  be a module over  $R(\Gamma)$ . If  $M$  is flat over any subring of the form  $R(\Lambda)$  where  $\Lambda$  is a finitely generated subgroup of  $\Gamma$  then  $M$  is flat over  $R(\Gamma)$ .*

As mentioned in the introduction, the proof of Theorem 1.1 is based on a theorem of Benson and Goodearl. For the reader's convenience we recall it here:

**Theorem** (Benson and Goodearl [4]). *Let  $R(\Gamma)$  be a strongly graded ring over  $R$  and let  $H$  be a subgroup of finite index of  $\Gamma$ . Let  $M$  be a flat module over  $R(\Gamma)$  which is projective over  $R(H)$ . Then  $M$  is projective over  $R(\Gamma)$ .*

**Proof of Theorem 1.1.** Let  $H$  be a subgroup of finite index of  $\Gamma$  and let  $R$  be any ring. Assume Moore's condition holds for the triple  $(\Gamma, H, R)$  and let  $M$  be a module over  $R(\Gamma)$ , projective over  $R(H)$ . We need to show  $M$  is projective over  $R(\Gamma)$ . Applying the above theorem of Benson and Goodearl it is sufficient to prove that  $M$  is flat over  $R(\Gamma)$ . By Lemma 2.1 it is sufficient to show that  $M$  is flat over  $R(\Lambda)$  for every finitely generated subgroup  $\Lambda$  of  $\Gamma$ . Let  $\Lambda$  be such a group and let  $H_\Lambda = H \cap \Lambda$ . Clearly,  $M$  is projective over  $R(H_\Lambda)$ . Furthermore, since Moore's condition holds for the triple  $(\Gamma, H, R)$  it holds also for the triple  $(\Lambda, H_\Lambda, R)$  and hence  $M$  is projective over  $R(\Lambda)$ . This implies that  $M$  is flat over  $R(\Lambda)$  and the result follows.  $\square$

Two important ingredients in the proof of the main theorem (Theorem 3.1) are Chouinard's theorem and Maschke's theorem for strongly graded rings. Since we will be using them repeatedly we recall them here starting with Chouinard's

**Definition 2.2.** Let  $\Gamma$  be a finite group. We say that an  $R(\Gamma)$ -module  $M$  is weakly projective if there is an  $f \in \text{Hom}_R(M, M)$  with  $\text{tr}_\Gamma(f) = \sum_{\sigma \in \Gamma} \sigma(f) = \text{id}_M$  (here  $\Gamma$  acts on  $\text{Hom}_R(M, M)$  diagonally).

**Theorem** (Chouinard [6], Aljadeff and Ginosar [1]). *Let  $R$  be an arbitrary ring with identity and let  $\Gamma$  be a finite group. Let  $R(\Gamma)$  be a strongly graded ring. If  $M$  is any module over  $R(\Gamma)$  then it is weakly projective (projective) if and only if it is weakly projective (projective) over all subrings  $R(P)$  where  $P$  is an elementary abelian subgroup of  $\Gamma$ . In fact it is sufficient to assume that  $M$  is projective over  $R(P)$  where  $P$  runs over representatives of all conjugacy classes of maximal elementary abelian subgroups of  $\Gamma$ .*

**Theorem** (Maschke [14, Chapter 1, Section 4]). *Let  $R$  be an arbitrary ring with identity and let  $\Gamma$  be a finite group whose order is invertible in  $R$ . Let  $R(\Gamma)$  be a strongly graded ring. Then any  $R(\Gamma)$ -module is weakly projective. In particular, any module  $M$  over  $R(\Gamma)$  is projective if and only if it is projective over  $R$ .*

- Note that Chouinard's and Maschke's theorems imply Moore's conjecture for  $(\Gamma, H, R)$  whenever the group  $\Gamma$  is finite.

In the proof of Theorem 3.1 it is convenient to replace  $H$  by a normal subgroup  $H_0$ . Let  $H$  be a subgroup of finite index in  $\Gamma$ . Let  $H_0$  be the core of  $H$  in  $\Gamma$  that is  $H_0 = \cap H^g$ .

**Lemma 2.3.** *Moore's condition holds for the triple  $(\Gamma, H, R)$  if and only if it holds for the triple  $(\Gamma, H_0, R)$ . Furthermore, if Moore's conjecture holds for the triple  $(\Gamma, H_0, R)$  then it holds for the triple  $(\Gamma, H, R)$ .*

**Proof.** If  $z$  is of order  $p$  and not in  $H_0$  then  $z$  is not in one of the conjugates  $H^g$  of  $H$ . Then  $z^{g^{-1}}$  which is of order  $p$ , is not in  $H$ . The second statement follows from the fact that a projective module over  $R(H)$  is projective over  $R(H_0)$ .  $\square$

It is natural to ask whether Moore's condition for a pair  $(\Gamma, H)$  is necessary in Moore's conjecture. More precisely we ask the following questions:

1. Assume Moore's condition does not hold for  $(\Gamma, H)$ . Is there a ring  $R$ , a strongly graded ring  $R(\Gamma)$  and a module  $M$  over  $R(\Gamma)$  which is projective over  $R(H)$  but not projective over  $R(\Gamma)$ ?
2. Let  $R$  be given and assume Moore's condition does not hold for  $(\Gamma, H, R)$ . Is there a module  $M$  over a strongly graded ring  $R(\Gamma)$  which is projective over  $R(H)$  but not over  $R(\Gamma)$ ?

**Proposition 2.4.** *If the group  $H$  is normal in  $\Gamma$  then Moore's condition is necessary in the following (strong) sense: If Moore's condition does not hold for  $(\Gamma, H, R)$  then there is a module  $M$  over  $R\Gamma$  (the group ring) which is projective over  $RH$  but not over  $R\Gamma$ .*

**Remark.** We cannot expect the proposition to hold for every strongly graded ring (e.g. the Gauss integers  $\mathbb{Z}[i]$  may be represented as a twisted group ring of the group of two elements over  $\mathbb{Z}$ . Moore's condition does not hold for  $(C_2, \{e\}, \mathbb{Z})$  but every module over  $\mathbb{Z}[i]$  which is projective over  $\mathbb{Z}$  is projective over  $\mathbb{Z}[i]$ ).

**Proof of Proposition 2.4.** Let  $\sigma$  be an element in  $\Gamma \setminus H$  of order  $p$  where  $p$  is not invertible in  $R$ . Denote by  $U$  the cyclic group it generates. Consider the left  $R\Gamma$ -module  $M = R\Gamma \otimes_{\mathbb{Z}U} \mathbb{Z}$  (with the obvious left  $R\Gamma$ -structure). The module  $M$  is free over  $RH$  with a basis consisting a set of representatives for the cosets of the subgroup  $\langle H, U \rangle$  in  $\Gamma$ . We claim that the map  $\pi : R\Gamma \rightarrow M$ ,  $\pi(g) = g \otimes 1$  does not split over  $R\Gamma$ . Indeed, if  $j$  is a splitting over  $R\Gamma$ , let

$$j(1 \otimes 1) = \sum_{i=0}^{p-1} \alpha_i \sigma^i + \sum_{i=0}^{p-1} \sum_{s_j \in T_{[\Gamma:U] \setminus \{1\}}} \alpha_{ji} s_j \sigma^i,$$

where  $T_{[\Gamma:U]}$  is a set of representatives for the cosets of  $U$  in  $\Gamma$  and such that  $1 \in T_{[\Gamma:U]}$ . By the splitting condition we get that

$$\sum_{i=0}^{p-1} \alpha_i = 1.$$

Furthermore, by the  $R\Gamma$ -linearity of the map  $j$  we get that  $\alpha_0 = \dots = \alpha_{p-1}$  and so  $p$  is invertible in  $R$ . Contradiction.  $\square$

The fact that the group  $H$  is normal in  $\Gamma$  is essential in Proposition 2.4. Next we give an example of a finite group  $\Gamma$  and a subgroup  $H$  such that  $\Gamma \setminus H$  contains an element of prime order, but for any ring  $R$ , any strongly graded ring  $R(\Gamma)$  and any module  $M$  over  $R(\Gamma)$ , if  $M$  is projective over  $R(H)$ , then it is projective also over  $R(\Gamma)$ .

**Example.** Let  $\Gamma = \langle \sigma, \tau : \sigma^9 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$  the dihedral group of order 18. Let  $H = \langle \sigma^3, \tau \rangle$ . The set  $\Gamma \setminus H$  contains elements of order 2 and hence Moore's condition does not hold for  $(\Gamma, H)$ . On the other hand every elementary abelian subgroup of  $\Gamma$  is cyclic and conjugate to a subgroup of  $H$ . The result follows from Chouinard's theorem.

We close this section by recalling some basic concepts and fixing notation concerning profinite topologies and profinite completions of groups (we refer the reader to [16]). A non-empty collection  $\Sigma$  of normal subgroups of finite index of a group  $\Gamma$  is filtered from below if for any  $N_1, N_2 \in \Sigma$  there exists  $N \in \Sigma$  such that  $N \subseteq N_1 \cap N_2$ . Then  $\Gamma$  turns into a topological group by considering  $\Sigma$  as a fundamental system of neighborhoods of the identity element 1 of  $\Gamma$ . We denote by  $K_\Sigma(\Gamma) = \lim_{\leftarrow} \Gamma/N$  the



profinite completion with respect to that topology and by  $\phi : \Gamma \rightarrow K_\Sigma(\Gamma)$  the canonical map induced by the natural projections  $\Gamma \rightarrow \Gamma/N$ ,  $N \in \Sigma$ .

Given any collection  $\Sigma$  as above let  $I_\Sigma$  be the index set that corresponds to  $\Sigma$ , that is for every  $N \in \Sigma$  we have  $i_N \in I_\Sigma$ . Obviously, using the ordering  $i_{N_1} > i_{N_2}$  if and only if  $N_1 < N_2$ , the set  $I_\Sigma$  is partially ordered and directed. For  $k < j$  in  $I_\Sigma$ , we denote by  $\phi_{jk} : \Gamma/N_j \rightarrow \Gamma/N_k$  the natural projection.

Finally, recall that if  $H$  is a subgroup of  $\Gamma$  of finite index and  $\Sigma$  is a collection as above with the additional condition that all elements  $N \in \Sigma$  are contained in  $H$ , then there is a natural inclusion  $K_\Sigma(H) \hookrightarrow K_\Sigma(\Gamma)$ .

### 3. Discrete groups

**Theorem 3.1.** *Let  $\Gamma$  be any group and  $H$  a subgroup of finite index. Let  $\Sigma$  be a collection of subgroups of  $H$ , filtered from below, normal and of finite index in  $\Gamma$ . Let  $K_\Sigma(\Gamma)$  and  $K_\Sigma(H)$  be the profinite completions of  $\Gamma$  and  $H$  with respect to  $\Sigma$ . Assume any element of prime order (say  $p$ ) in  $K_\Sigma(\Gamma) \setminus K_\Sigma(H)$  is conjugate to an element  $\phi(x)$  where  $x$  is an element in  $\Gamma$  of order  $p$ . Then Moore's conjecture holds for  $(\Gamma, H)$ . In particular if  $K_\Sigma(\Gamma)$  is torsion free then Moore's conjecture holds for  $(\Gamma, H)$ .*

**Proof.** Let  $H_0$  be the core of  $H$  in  $\Gamma$ . By Lemma 2.3 it is sufficient to show the conjecture for  $(\Gamma, H_0)$ . Note that since the groups in  $\Sigma$  are normal in  $\Gamma$ , they are contained in  $H_0$ . For every  $j \in I_\Sigma$ , we denote by  $\phi_j : \Gamma/N_j \rightarrow \Gamma/H_0$  the natural projection.

For every  $j \in I_\Sigma$ , let  $Y_j$  be the subset of  $\prod_{i \in I_\Sigma} \Gamma/N_i$  defined by

$$Y_j = \{(x_i) \in \prod_{i \in I_\Sigma} \Gamma/N_i : \phi_{jk}(x_j) = x_k \text{ whenever } k < j\}.$$

For any prime number  $p$ , let  $Z_j^{(p)} \subset Y_j$  be the set

$$Z_j^{(p)} = \{(x_i) \in Y_j : \text{ord}(x_j) = p \text{ in } \Gamma/N_j \text{ and } \text{ord}(\phi_j(x_j)) = p \text{ in } \Gamma/H_0\}.$$

Note that  $Z_j^{(p)} \supset Z_{j'}^{(p)}$  for  $j < j'$ .

The main step of the proof is the first statement in the following lemma. Recall that by definition Moore's conjecture holds for  $(\Gamma, H_0)$  if and only if Moore's conjecture holds for  $(\Gamma, H_0, R)$  with  $R$  arbitrary.  $\square$

**Lemma 3.2.** *If the conclusion in Moore's conjecture does not hold for  $(\Gamma, H_0, R)$  (that is there is a non-projective  $R(\Gamma)$ -module  $M$  which is projective over  $R(H_0)$ ), then there is a prime number  $p$ , not invertible in  $R$ , such that  $Z_j^{(p)}$  is non-empty for every  $j \in I_\Sigma$ . Furthermore,  $Z_j^{(p)}$  is closed in  $\prod_{i \in I_\Sigma} \Gamma/N_i$ .*

Let us postpone the proof of the lemma and complete first the proof of the theorem. Assume the theorem is false. This means that Moore's conjecture does not hold for a triple  $(\Gamma, H_0, R)$ , some  $R$ . In particular the conclusion in Moore's conjecture does not hold for  $(\Gamma, H_0, R)$ . Let  $p$  be the prime number given by the lemma. From the condition  $Z_j^{(p)} \supset Z_{j'}^{(p)}$  for  $j < j'$  it follows that the family  $\{Z_j^{(p)}\}_{j \in I_\Sigma}$  satisfies the finite intersection property. Moreover, since  $\prod_{i \in I_\Sigma} \Gamma/N_i$  is compact and the sets  $Z_j^{(p)}$ 's are closed, there is an element  $z \in \bigcap_{i \in I_\Sigma} Z_i^{(p)}$ . Clearly,  $z$  is an element of order  $p$  in  $K_\Sigma(\Gamma)$ . Furthermore, by the definition of the sets  $Z_j^{(p)}$ , we have  $z \notin K_\Sigma(H_0)$ . Now, it is easily checked that the condition (in the theorem) on elements of  $K_\Sigma(\Gamma) \setminus K_\Sigma(H)$  holds for all elements in  $K_\Sigma(\Gamma) \setminus K_\Sigma(H_0)$  and hence there is an element  $x$  in  $\Gamma$  of order  $p$  such that  $\phi(x)$  is conjugate to  $z$  (in  $K_\Sigma(\Gamma)$ ). It is clear that  $x \notin H_0$  and we get a contradiction to Moore's condition (M1) or (M2) for  $(\Gamma, H_0, R)$ .

**Proof of Lemma 3.2.** Let  $M$  be an  $R(\Gamma)$ -module, projective over  $R(H_0)$  but not projective over  $R(\Gamma)$ . Write  $R(\Gamma) = R(H_0)(\Gamma/H_0)$ . By Chouinard's and Maschke's theorems there is a prime number  $p$ , not invertible in  $R$ , and an elementary abelian  $p$ -group  $E$  in  $\Gamma/H_0$  such that  $M$  is not projective over the ring  $R(H_0)(E)$ . Let  $T$  be the inverse image of  $E$  in  $\Gamma$  (with respect to the group extension  $1 \rightarrow H_0 \rightarrow \Gamma \rightarrow \Gamma/H_0 \rightarrow 1$ ). Now, for any  $N_j \in \Sigma$  write  $R(T) = R(N_j)(T/N_j)$ . Since  $N_j \subset H_0$ ,  $M$  is projective over  $R(N_j)$ . Furthermore,  $M$  is not projective over  $R(T) = R(H_0)(E) = R(N_j)(T/N_j)$  and hence by Chouinard's theorem there is an elementary abelian subgroup  $E_j$  of  $T/N_j$  such that  $M$  is not projective over  $R(N_j)(E_j)$ . Let  $T_j$  be the inverse image of  $E_j$  in  $T$ . The module  $M$  is not projective over  $R(T_j)$  and therefore  $T_j$  is not contained in  $H_0$ . We conclude that  $T_j/N_j = E_j$  is elementary  $p$ -abelian with nontrivial image modulo  $H_0$ . This proves the first statement of the lemma. For the second statement in the lemma observe that the set  $Z_j^{(p)}$  contains  $\Gamma/N_i$  for all  $i$  but a finite subset of  $I_\Sigma$ . This completes the proof of the lemma and hence of Theorem 3.1.  $\square$

We close this section with the

**Proof of Theorem 1.2.** Assume Moore's condition holds for  $(\Gamma, H, R)$  where  $H$  is a subgroup of finite index of  $\Gamma$  and  $R$  is any ring. By Lemma 2.3 it is sufficient to prove that Moore's conjecture holds for  $(\Gamma, H_0, R)$ . Let  $\Omega_\Gamma(H_0) = \{H_0 \cap N : N \in \Omega_\Gamma\}$ . Since  $K_{\Omega_\Gamma(H_0)}(\Gamma) \leq K_{\Omega_\Gamma}(\Gamma)$  (in fact they are naturally isomorphic) the condition in Theorem 3.1 (for elements in  $K_{\Omega_\Gamma(H_0)}(\Gamma) \setminus K_{\Omega_\Gamma(H_0)}(H_0)$ ) holds and the result follows.  $\square$

#### 4. Profinite groups, complete group rings and profinite modules

Let  $\Gamma$  be a profinite group and  $H$  an open subgroup of  $\Gamma$ . We denote by  $T_{[\Gamma:H]}$  a transversal for the right cosets of  $H$  in  $\Gamma$  (we choose  $1 \in \Gamma$  as the representative of the trivial coset). Let  $M$  be a profinite module over the complete group ring  $[[R\Gamma]]$ . We say that  $M$  is  $([[R\Gamma]], [[RH]])$ -relative projective if given any diagram of  $[[R\Gamma]]$ -modules

and  $[[R\Gamma]]$ -continuous maps  $(\alpha, \phi)$

$$\begin{array}{ccccc} & & M & & \\ & \swarrow & \downarrow \phi & & \\ B & \xrightarrow{\alpha} & A & \longrightarrow & 0 \end{array}$$

then if  $\phi$  can be lifted to a continuous map  $M \rightarrow B$  over  $[[RH]]$  then it can be lifted also to a continuous map  $v : M \rightarrow B$  over  $[[R\Gamma]]$ .

Maschke's theorem in this context reads:

**Theorem 4.1.** *If  $[\Gamma : H]$  (the index of  $H$  in  $\Gamma$ ) is invertible in  $R$  then every profinite module  $M$  over  $[[R\Gamma]]$  is  $([[R\Gamma]], [[RH]])$ -relative projective.*

**Proof.** With the above notation, if  $\theta_H : M \rightarrow B$  is a continuous lifting of  $\phi$  over  $[[RH]]$  then  $\text{ord}(T_{[\Gamma:H]})^{-1} \sum_{\sigma \in T_{[\Gamma:H]}} \sigma \theta_H \sigma^{-1}$  is a continuous lifting of  $\phi$  over  $[[R\Gamma]]$ .  $\square$

The following lemma may be seen as a profinite version of Higman's criterion (see [3, Proposition 3.6.4]):

**Lemma 4.2.** *With the above notation, a module  $M$  is  $([[R\Gamma]], [[RH]])$ -relative projective if and only if there is an  $[[RH]]$ -continuous map  $s : M \rightarrow M$  such that  $\text{tr}_{H \rightarrow G}(s) = \sum_{\sigma \in T_{[\Gamma:H]}} \sigma(s) = \text{id}_M$  (where  $\sigma$  acts on  $\text{Hom}_{[[RH]]}(M, M)$  diagonally).*

**Proof.** Consider the induced module  $[[R\Gamma]] \hat{\otimes}_{[[RH]]} M$  of  $M$  where  $\hat{\otimes}$  denotes the complete tensor product (see [16, Section 5.5]). Since  $[[R\Gamma]]$  is finitely generated as an  $[[RH]]$ -module, the complete tensor product coincides with the usual tensor product  $[[R\Gamma]] \otimes_{[[RH]]} M$ .

Assume  $M$  is  $([[R\Gamma]], [[RH]])$ -relative projective and consider the diagram of  $[[R\Gamma]]$ -modules

$$\begin{array}{ccccc} & & M & & \\ & \swarrow & \downarrow \text{id} & & \\ [[R\Gamma]] \otimes_{[[RH]]} M & \xrightarrow{\pi} & M & \longrightarrow & 0 \end{array}$$

where  $\pi$  is given by  $\sigma \otimes m \rightarrow \sigma m$ . The map  $\pi$  splits over  $[[RH]]$  ( $j : m \rightarrow 1 \otimes m$  is a splitting) and so it splits over  $[[R\Gamma]]$ . Let  $v : M \rightarrow [[R\Gamma]] \otimes_{[[RH]]} M$  be a splitting of  $\pi$  over  $[[R\Gamma]]$ . Since the elements of  $T_{[\Gamma:H]}$  form a basis of  $[[R\Gamma]]$  over  $[[RH]]$  we may write

$$v(m) = \sum_{\sigma \in T_{[\Gamma:H]}} \sigma \otimes f_\sigma(m),$$

where  $f_\sigma : M \rightarrow M$  is a well-defined map.

Now, one checks that the map  $f_\sigma$  is  $R$ -linear and moreover  $f_\sigma(hm) = \sigma^{-1}h\sigma f_\sigma(m)$  for every  $h \in H$  and  $m \in M$ . In particular  $f_e : M \rightarrow M$  is an  $RH$ -linear map. We show that  $f_e$  is continuous (and hence  $[[RH]]$ -linear). Let  $M_0 \subset M$  be an open  $[[RH]]$ -submodule. We claim that  $f_e^{-1}(M_0)$  contains an open  $[[RH]]$ -submodule of  $M$ . To see this, observe that  $[[R\Gamma]] \otimes_{[[RH]]} M_0$  is open in  $[[R\Gamma]] \otimes_{[[RH]]} M$  and by the continuity of  $\nu$  we have that  $\nu^{-1}([R\Gamma] \otimes_{[[RH]]} M_0) = M_1$  is open in  $M$ . By the linear independence of the elements in  $T_{[\Gamma:H]}$  over  $[[RH]]$  we have that  $f_e(M_1) \subset M_0$  and the claim is proved. Finally we show that  $tr_{H \rightarrow G}(f_e) = id_M$ . By the  $\Gamma$ -linearity of  $\nu$  we obtain that  $f_\sigma(m) = f_e(\sigma^{-1}m)$  and from the equality  $\pi\nu = id_M$  we obtain  $tr_{H \rightarrow G}(f_e) = \sum_{\sigma \in T_{[\Gamma:H]}} \sigma f_e \sigma^{-1} = id_M$ .

For the converse let  $f : M \rightarrow M$  be a continuous  $[[RH]]$ -map such that  $tr_{H \rightarrow \Gamma}(f) = id_M$ . Let

$$\begin{array}{ccccc} & & M & & \\ & \theta_H \swarrow & \downarrow \phi & & \\ B & \xrightarrow{\alpha} & A & \longrightarrow & 0 \end{array}$$

be a diagram of  $[[R\Gamma]]$ -modules where  $\phi$  and  $\alpha$  are  $[[R\Gamma]]$ -continuous maps and  $\theta_H$  an  $[[RH]]$ -continuous map. The map  $\theta_\Gamma = tr_{H \rightarrow \Gamma}(\theta_H f)$  is an  $[[R\Gamma]]$ -continuous map and it is a lifting of  $\phi$ . The lemma is proved.  $\square$

Assume now,  $H$  is normal (and open) in  $\Gamma$ . Denote by

$$Hom_{[[RH]]}(M, M) = End_{[[RH]]}(M)$$

the endomorphism ring of all continuous  $[[RH]]$ -linear homomorphisms  $\phi : M \rightarrow M$ . The quotient group  $\Gamma/H$  acts on  $End_{[[RH]]}(M)$  via the diagonal action. Recall that a  $\Gamma/H$ -module  $U$  is weakly projective if and only if there is an additive map  $\phi : U \rightarrow U$  with  $\sum_{\sigma \in \Gamma/H} \sigma \phi \sigma^{-1} = id_U$ .

**Proposition 4.3.** *Let  $\Gamma$  be a profinite group,  $H$  a normal open subgroup of  $\Gamma$ ,  $R$  a profinite ring and  $M$  a profinite  $[[R\Gamma]]$ -module. Then the following conditions are equivalent:*

1.  $M$  is  $([[R\Gamma]], [[RH]])$ -relative projective.
2. There exists a continuous  $[[RH]]$ -map  $f : M \rightarrow M$  with  $tr_{H \rightarrow \Gamma}(f) = id_M$ .
3.  $Hom_{[[RH]]}(M, M)$  is weakly projective  $\Gamma/H$ -module (with the diagonal action).
4.  $Hom_{[[RH]]}(M, M)$  is weakly projective  $E$ -module for every elementary abelian subgroup  $E$  of  $\Gamma/H$ .

**Proof.** The equivalence of (1) and (2) is in Lemma 4.2.

(2  $\rightarrow$  3) Let  $f : M \rightarrow M$  be a continuous  $[[RH]]$ -map with  $\sum_{\sigma \in \Gamma/H} \sigma(f) = id_M$ . Applying the functor  $Hom_{[[RH]]}(-, M)$  to  $f$  we obtain a map  $f^* : Hom_{[[RH]]}(M, M) \rightarrow Hom_{[[RH]]}(M, M)$  of abelian groups such that  $\sum_{\sigma \in \Gamma/H} \sigma f^* \sigma^{-1} = id_{Hom_{[[RH]]}(M, M)}$ .

(3  $\rightarrow$  2) Let  $\phi : \text{Hom}_{[[RH]]}(M, M) \rightarrow \text{Hom}_{[[RH]]}(M, M)$  be a map of abelian groups such that  $\sum_{\sigma \in \Gamma/H} \sigma \phi \sigma^{-1} = \text{id}_{\text{Hom}_{[[RH]]}(M, M)}$ . Then  $\phi(\text{id}_M) : M \rightarrow M$  is a continuous  $[[RH]]$ -map with  $\sum_{\sigma \in \Gamma/H} \sigma(\phi(\text{id}_M)) = \text{id}_M$ .

(3  $\leftrightarrow$  4) (see [1, Theorem 1]).  $\square$

**Theorem 4.4.** *Let  $H$  be an open normal subgroup of a profinite group  $\Gamma$ . Let  $M$  be a profinite  $[[R\Gamma]]$ -module. Then  $M$  is projective over  $[[R\Gamma]]$  if and only if  $M$  is projective over  $[[RT]]$  for every  $T \leq \Gamma$  which contains  $H$  and  $T/H$  is elementary abelian.*

**Proof.** Observe that an  $[[R\Gamma]]$ -module  $M$  is projective if and only if it is projective over  $[[RH]]$  and  $([[R\Gamma]], [[RH]])$ -relative projective. The theorem now follows from Proposition 4.3 implications (1  $\rightarrow$  3) for  $(T, H)$ , (4  $\rightarrow$  3), (3  $\rightarrow$  1) for  $(\Gamma, H)$ .  $\square$

We can prove now our main theorem for profinite groups.

**Proof of Theorem 1.6.** By taking the core of  $H$  in  $\Gamma$  we can assume  $H$  is normal in  $\Gamma$ . Let  $\Omega_\Gamma$  denote the family of all normal, open subgroups of  $\Gamma$  and let  $\Omega_\Gamma(H) = \{H \cap N : N \in \Omega_\Gamma\}$ . Clearly, the family  $\Omega_\Gamma(H)$  is cofinal in  $\Omega_\Gamma$  and hence the completions of  $H$  and  $\Gamma$  with respect to  $\Omega_\Gamma(H)$  are naturally isomorphic to  $H$  and  $\Gamma$  respectively. In particular the condition in Theorem 3.1 is satisfied in an obvious way. The proof of the theorem is completed by following the steps in the proof of Theorem 3.1. The only place here which is slightly different is in the proof of Lemma 3.2. Here instead of using strongly graded rings, we apply Theorem 4.4. Details are left to the reader.  $\square$

**Final Remark.** The original proof of Serre's theorem on cohomological dimensions of profinite groups, which was quoted in the Introduction, uses Serre's result on products of Bockstein operators [17]. It is not surprising that Chouinard's theorem (which is the main ingredient in our proof) is based also on Serre's result on Bockstein operators.

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